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Concreteness Fading of Algebraic Instruction: Effects on Learning

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Abstract

Learning algebra is difficult for many students, in part due to an emphasis on the memorization of abstract rules. Algebraic reasoners across expertise levels often rely on perceptual-motor strategies to make these rules meaningful and memorable. However, in many cases, rules are provided as patterns to be memorized verbally, with little overt perceptual support. Although most work on concreteness focuses on conceptual support through examples or analogies, we here consider *notational concreteness*—perceptual/motor supports that provide access into the dynamic structure of a representation itself. We hypothesize that perceptual support may be maximally beneficial as an initial scaffold to learning, so that later static symbol use may be interpreted using a dynamic perspective. This hypothesis meshes with other findings using concrete analogies or examples, which often find that fading these supports over time leads to stronger learning outcomes. In an experiment exploring this hypothesis, we compare gains from the fading out of dynamic concrete physical motion of symbols during instruction with the introduction of motion over the course of instruction. In line with our theoretical perspective,

concreteness fading led to significantly higher achievement than concreteness introduction after Day 2 of the intervention.

Algebraic notation is a foundational domain for understanding all areas of advanced mathematics, for developing critical thinking skills, and for preparation in a wide variety of careers. Despite the emphasis on algebra during middle and high school, many students struggle to achieve a basic understanding of algebraic formalisms such as equations (Koedinger, Alibali, & Nathan, 2008). In the 2011 National Assessment of Educational progress, 27% of eighth graders scored below the basic level in mathematics and only 34% reached proficient or advanced levels (National Center for Education Statistics, 2011).

Much difficulty in algebra learning stems from failure to achieve a robust, fluent, procedural understanding of the structure, arrangement, and legal transformations of an algebraic expression. Many students struggle to understand the basic principles of algebraic notation when they encounter them (Bernardo & Okagaki, 1994; Martin & Bassok, 2005), have difficulty converting between formal expressions and other representations and real situations (Koedinger & Nathan, 2004; Lochhead & Mestre, 1988), and often don't know what transformations are legal and appropriate (Marquis, 1988). The National Mathematics Advisory Panel concluded in 2008 that "[Many students] have difficulty grasping the syntax or structure of algebraic expressions and do not understand procedures for transforming equations or why transformations are done the way they are". Students often memorize abstract rules, which may not lead to the internalization of algebraic notation and may cause students to overgeneralize rules because of visual similarity (Kirshner & Awtry, 2004; Marquis, 1988). The panel concluded, furthermore, that students have difficulty simply reading, writing, and transforming formal expressions.

Because advanced topics such as physics, statistics, and calculus often make extensive use of algebraic notation, these struggles make it almost impossible to understand advanced STEM concepts, and these difficulties are compounded as equations become more complex.

We can generally distinguish among multiple kinds of inscriptions people make in the course of mathematical reasoning. We are particularly interested in *formal inscriptions*, or *formalisms*, which we take to be conventional inscriptions using a fixed alphabet of symbols, and well-defined rules of interpretation, as well as a clearly defined system for instantiating variables and substitution. Formalisms play a critical role in algebraic reasoning, beginning especially with the most famous formalism, algebraic notation—the familiar system of equations, expressions, variables, and operators. Although formal manipulations are only part of understanding algebra, and although many concepts may be best initially learned when taught without formal inscriptions using examples and concrete reasoning (Nathan, 2012), equations continue as a core representation across the STEM disciplines.

Given its importance and the difficulties that students exhibit, it is important to explore new ways to approach instruction and pedagogy related to teaching algebraic structure. Our general approach is to explore the role of explorable, dynamic symbols rendered in computer algebra systems. At this point, it is not clear whether such tools can be useful methods for imparting mathematics at all, and, if so, how they may best be used in conjunction with traditional symbolic instruction. As such, the primary aim of this study is to examine the effects of Pushing Symbols, a novel approach to teaching algebraic structure that utilizes perceptual-motor routines, on mathematics learning. In particular, we contrast two situations: one in which dynamic symbols precedes traditional instruction on a particular topic, and another in which they follow it.

Perceptual-Motor Routines and Mathematics Understanding

Substantial empirical evidence demonstrates that perceptual-motor routines are also an important component of algebraic reasoning. For example, Ullman (1984) articulated the idea that visual processing frequently relies on assemblages of a limited set of 'visual routines': shifting attention to a region or object, mentally rotating or scaling a space, tagging elements for later tracking, grouping elements according to gestalt principles or deliberate action, and so on. This idea has been applied many times to mental rotation, diagrammatic reasoning (Bannerjee & Chandrasekharan, 2004), perception in the world (Hayhoe 2000; Pylyshyn, 2000; Rao & Ballard, 1995), and visual search (Horswill, 1995). Though visual routines were articulated first and have been the focus of most attention, similar approaches can be applied to motor programs and other sensory systems; thus we here use the general term *perceptual-motor routines*.

In applying this concept to formal algebra, we further leverage the notion of neural reuse (Anderson, 2014, 2015; Dehaene, 2007). The general idea of neural reuse is that skills and abilities that are typically associated with one mental domain or content may be applied to a new domain over cultural or evolutionary time scales, especially when cultural tools facilitate that cooption (Anderson, 2014; Landy, Allen & Anderson, 2011). On this account, part of the power of formal systems is the way they typically afford perceptual-motor routines: routines that deploy or instantiate conceptual and procedural knowledge, but which are themselves coded in perceptual-motor processing systems.

Substantial empirical work has demonstrated that notation reading and manipulation involves not just memorizing explicit rules, but also learning appropriate perceptual routines (Goldstone, Landy, & Son, 2010; Kirshner, 1989; Maruyama, Pallier, Jobert, Sigman, & Dehaene, 2012). Successful reasoners in mathematics often view algebraic expressions as structured, physical objects (Dörfler, 2006; Kellman, Massey, & Son, 2010; Landy & Goldstone, 2007a; Radford & Puig, 2007; Wittgenstein, 1922), or depictions of such objects, and use visual routines over those perceptions to solve mathematical problems. Practically, to understand algebraic notation, students should visualize expressions as composed of multiple parts or objects (CCSSI-M, 2010) and categorize these objects based on their mathematical functions. In addition to this object-based metaphor, rigid motion is a powerful perceptual grouping mechanism that could be a strong tool if engaged while learning algebraic structure (Kilpatrick, Swafford, & Findell, 2001). Motion and visual patterns are intuitive in mathematics (Dörfler, 2006). Students may better recognize the construction of algebraic notation if they dynamically transform expressions using physical manipulation and perceptual training (Kellman et al., 2008; Kirshner & Awtry, 2004; Maruyama et al., 2012). Indeed, this pictorial approach to algebra seems to have its roots in the very beginnings of algebraic notation (Heeffer, 2013).

Although it is perhaps common to think of a mathematical derivation as a sequence of written elements, perceptual-motor routines that act on these written elements normally track motion over time. We refer to these processes that normally track continuous real-world motion as *dynamic processes*, and to systems that instantiate that motion as *dynamic* interactions. An apt analogy is that we treat standard mathematical notation as something like a comic strip, which uses static positions at different time points to represent an underlying continuous process. That process is itself dynamic, as would be a physical model of that process (a film, composed of a large number of static images which are perceived continuously, perhaps represents an intermediate case).

Consider the brief derivation shown in Figure 1. There are two reasonable perspectives (at least!) on this derivation: one can certainly treat it as a sequence of four lines, which are licensed by particular entailment relationships. That is, given that the first line is true, the second line is also true. Given the first two lines, the third follows, etc. In this sense, the relationship between one line and the next is one of truth of the full utterance. However, one can also see this derivation as a sequence of actions taken over objects, sub served by particular perceptual routines. Getting from the second to the third line, for instance, involves the translation of the "+3y" right ward, and the transformation of the "+" into a "-". The third line turns into the fourth when the "3y" and the "-3y" meet and annihilate, leaving behind a 0. This conceptualization is ubiquitous in our language, in which we talk about "moving the 3y to the other side", "changing the sign", or "canceling" terms. The perceptual-motor routines perspective underlying our work suggests, further, that the actual processes implementing this sort of reasoning are often themselves perceptual in character.

Concreteness Fading versus Concreteness Introduction

Of interest to this line of research is the comparative utility of concrete and abstract instructional designs when introducing algebraic concepts and formalisms. Traditionally, formalisms are often introduced early and tend to be followed by examples, even though for learners this path is often fraught and difficult. The current research focuses on formalisms, but uses two different approaches to teach them. One is akin to the traditional *formal* approach, while the other (which we'll call the *concrete* approach) focuses on using perceptual-motor training and gestures (in the user interface sense) to teach the symbols-as-objects framework. Although the formal approach is most common, it is plausible that once a symbols-as-objects framework has been thoroughly internalized, students may be better able to apply explicitly stated rules and formalisms. Experts and novices in mathematics have consistently been shown to treat symbols as physical, both in their metaphorical language (Marghetis & Núñez, 2013; Nogueira de Lima & Tall, 2008), and also in the kinds of psychological processes used to transform and parse them (Kirshner, 1989; Landy & Goldstone, 2007b; Marquis, 1988; Maruyama et al., 2012). We take it as assumed that both formal and concrete approaches have some plausible value, and consider here the relative order of introduction.

Concrete instruction that integrates perceptual experiences of new concepts may aid learning by activating real-world knowledge structures and increasing learner engagement (Goldstone & Son, 2005; McNeil & Fyfe, 2012). However in some cases, abstract instruction, which utilizes more arbitrary referents, may be more beneficial in generalizing concepts to novel systems, perhaps by removing the distracting or context-specific elements of concrete materials that may be irrelevant to the underlying concept (Goldstone & Sakamoto, 2003; Kaminski, Sloutsky & Heckler, 2009). A possible solution to this paradox is the process of *concreteness fading*—a shifting from concrete to abstract representations in instruction (Bruner, 1966; Fyfe, McNeil, Son, & Goldstone, 2014; Goldstone & Son, 2005; McNeil & Fyfe, 2012). Gradually removing concrete scaffolding and introducing abstract concepts has predicted better achievement and retention than entirely abstract or concrete instruction in scientific (Goldstone & Son, 2005) and mathematical domains (McNeil & Fyfe, 2012).

In this study, we compare relatively concrete and abstract representations of algebraic notation. In algebraic notation, the dynamic nature of symbols is only partially captured by a proof or derivation—static forms are written in a sequence that reflects a temporal ordering. We examine how the order of presenting concrete perceptually-based and static abstract instruction might benefit student achievement, retention, and engagement in algebra. We instantiate concreteness fading and introduction as whether the reified metaphors of the object-based representation of mathematical notation are either presented concretely initially and later followed with static representations (*fading)*, or presented after more abstract representations (*introduction*). The "concreteness introduction" approach aligns with the "formalisms-first" approach that is commonly used in school algebra classes (Nathan, 2012); symbolic rules and transformations are taught first, and perceptual instantiations of those manipulations follow.

A coordinating perspective is that of preparation for future learning (PFL) (Bransford & Schwartz, 1999). The PFL perspective suggests that when given opportunities to explore and generate their own ideas about mathematical concepts in new environments before they are presented with more traditional algorithms and content, students can lay conceptual groundwork which while not helpful on its own, facilitates learning from more traditional lessons. A primary principle of PFL when applied to math and science domains is drawing attention to the underlying structure of the mathematics (Schwartz, Chase, Oppezzo, & Chin, 2011). Providing dynamic and concrete materials that students can explore when first being introduced to the material and then later removing them and replacing it with more abstract representation using algorithms and symbols may enable students to identify and focus in on the underlying mathematical structures and features. Our ‘concreteness fading’ condition aligns well with a PFL approach: dynamical symbols (described in detail in the next section) are used first, and the rules of manipulation are learned formally over static forms second. To be clear, however, no algebra approach, including the so-called “formalisms first” approach, typically employs dynamic symbols in the way we do. Because of the novelty of this approach, it is probably best to be careful translating our results to typical experiments testing the PFL theory or even to other instances of concreteness fading. Our intention in this project is to understand better how learning works in the context of dynamic symbols, rather than to test the general framework of PFL or concreteness fading per se.

Pushing Symbols

Consistent with the perceptual object-based framework discussed earlier, a concrete (yet formal) algebra intervention, *Pushing Symbols* (PS) was developed to embed the symbolism of algebra into core perceptual systems in a consistent and productive way (Ottmar, Landy, & Goldstone, 2012; 2015). By reifying mathematical symbols as movable physical objects, the intervention aims to help students identify algebraic structure, think more flexibly about expressions, and realize that transformations can be more dynamic than the static re-copying of lines.

If transformations of static symbols are internally realized through perceptual-motor simulation of continuous motion, the goal of Pushing Symbols is to literally present that continuous motion. Commutativity is realized by physically moving one symbol to the other side of another; when factoring 2x+3x into (2+3)x, the common symbols are literally joined and placed on top of each other, and put to the right-hand side. In each case, the user initiates some action, and that action is completed by automatic responses generated by the program (see Figure 2). This visualization is intended to literally depict the operation of plausible perceptual routines, and in this way to more closely reflect the perceptual routines involved in expert reasoning.

In the PS system, motion is concretely presented to the learner, and the learner actively engages in manipulating formal expressions with their hands. In PS, students discover dynamical versions of formal operations and practice algebraic principles by physically touching objects and representations of notation and moving symbols embedded in equations. For example, when simplifying the expression 4x+5+2x+3, students can touch the 5 with their finger, pick it up and move it to the left of the 3. When they release, it will result in the expression of 4x+2x+5+3. If they try to evaluate the middle sum (by tapping on the '+' sign), the system will refuse to combine 2x+5. If they tap on either of the other two (a legal operation), the corresponding operation is completed. In this way, PS affords learning the learning of algebra through active exploration and discovery; in the more structured contexts used here, this feature gives students fast feedback regarding common errors.

In theory, PS may help build students' understanding of algebraic concepts by internalizing the appropriate way of visualizing patterns in algebraic structure. One plausible mechanism is that moving symbols may help the construction of perceptual routines that implement formal transformations. It is also plausible that building perceptual routines that implement formal transformations will reduce cognitive load (Goldstone, Landy, & Son, 2010;

Landy, Allen, & Zednik, 2014). Taken together, this PS intervention may make subsequent formal/conceptual instruction more effective. However, to date, no research has been conducted to specifically test this theory or evaluate the feasibility or effectiveness of this intervention or other dynamic technologies.

In the Pushing Symbols approach, visual routines implement much mathematical and formal reasoning. This process of constructing visual routines serves to offload rule-based manipulation that is often verbally mediated, reducing cognitive load (Landy, Allen, & Zednik, 2014; Maruyama et al., 2012). We predict that this reduction will make it easier to acquire new formal information: that is, if visual processes are explicitly taught early, rather than discovered later, overall learning will be facilitated.

The PS intervention consists of several classroom instructional activities that aim to teach core algebraic concepts, such as distributivity, commutativity, and solving equations, as well as basic notational skills, such as the order of operations and simplifying expressions. The intervention also aims to use engaging instructional methods and technological tools to increase overall performance and interest in the field of mathematics. (Additional details and examples of the PS approach are described later in the methods section).

The Present Study

This study aims to compare whether utilizing dynamic concreteness fading within the Pushing Symbols intervention is more effective than dynamic concreteness introduction for teaching algebraic structure. We manipulated whether students engaged in a dynamic motion-based lesson with technology before a static lesson (concreteness fading) or a static lesson preceded a dynamic motion-based lesson (concreteness introduction). We hypothesize that concreteness fading of motion-based instruction will improve student performance more than the concreteness introduction condition.

Method

Participants

Participants were 98 7th grade students (43.9% male, 56.1% female) from a middle school located in a suburban area in the east coast. Students participated in our study during their regular math class time. All students obtained parental consent to participate in the study.

Materials and Procedures

In this research design, two intervention forms were compared. Students within classes were randomly assigned into one of two conditions: concreteness fading (dynamic first) or concreteness introduction (static first). There was no control group per se, since both groups received both the dynamic and static lessons. Rather, the difference between groups was due to the order and day in which the two lessons were given to students (whether dynamics were faded or introduced). The *concreteness fading* condition received the dynamic lesson on Day 1 and the static lesson lacking explicit symbol motion on day 2. The *concreteness introduction* condition received the static lesson on Day 1 and the dynamic lesson on day 2.

The study took approximately 3 hours and occurred over two class periods. On the first day, all students were asked to complete a pre-test, including 18 problems on simplifying algebraic expressions and a mathematics anxiety and self-efficacy questionnaire. After the pre-test, students were randomly assigned into two groups and separated into two different rooms based on their assigned condition. Students in the fading condition received the dynamic lesson first, while students in the introduction condition received the static lesson first. The content of the lessons and the time spent on each instructional activity were matched, however, the pedagogical practices used in each lesson differed, as described below. At the end of the first day, students took an 18-item post-test assessment that evaluated how much they learned during the class period and completed an engagement questionnaire.

On the second day, the students received the opposite lesson and took a third assessment. At this crucial time point of comparison (achievement after day 2), both conditions had received both the dynamic and static lessons. Students also completed a second engagement questionnaire. A month later, a fourth assessment was given to see how well the students retained the information. This retention assessment was identical in form to the other assessments.

Dynamic Lesson.

The dynamic lesson began with a 15-minute whole-group lesson on combining like terms, taught by the researchers. Dynamic instruction emphasized the commutative property and taught students how to physically move around and transpose terms into equivalent expressions. During the lesson, a researcher demonstrated how to identify and combine like-terms using color-coded magnets to represent mathematical principles such as numbers, symbols, variables and coefficients. These were designed to provide perceptual distinctions that aligned with syntactically and transformationally important structures—in this case, like terms. After the whole group lesson, students broke into small groups and explored the ideas discussed in the lesson by simplifying a series of expressions using colored manipulative tiles that matched the magnet system used in the whole group lesson. In this activity, students were asked to manipulate the expressions by rearranging the terms and replacing the tiles with equivalent terms after performing operations (see Figure 3 for an example).

Next, students were given iPads and asked to solve 120 problems on using the Pushing Symbols (PS) iPad application over 30 minutes. The PS iPad app provides fast feedback and engages perceptual-motor systems through fluid transformations as students touch the screen to manipulate expressions (see Figure 2 for sample problem transformations using PS). For example, students could touch operator signs to perform calculations and drag terms to different places in the expression. The gestures and actions that controlled user interactions were designed to be reminiscent of naturally produced gestures and diagrams. Unlike the tiles system, students received immediate feedback on the accuracy of their manipulation. The program did not allow students to make mistakes- when an error was made, the screen shook at them and required them to try again. Students were introduced to the interface through a brief guided problem set. Once they completed this introductory level, they were then free to move through the application at their own pace. The program included 10 levels with 12 problems increasing in complexity within each level. Students received between 1-3 stars for each problem that they completed based on accuracy and speed. Once students received 80% of the stars on a level, the next level of 12 problems was unlocked.

Static Lesson.

The static lesson used traditional methods with worked-examples and hand-written solutions. During the 15-minute whole group lesson, a researcher taught the targeted content of the commutative property and combining like terms using algorithms, static lines, and worked examples. Instead of colored magnets, a white board was used to demonstrate how to combine like terms and simplify the expression using static lines of written expressions similar to a proof. No motion or transposition was used in this lesson. After the lessons, students were broken into small groups and completed an activity using uncolored tiles to model the expression and demonstrate how to best simplify by performing operations. Rather than physically moving the tiles, students replaced tiles one operation at a time line by line to demonstrate their procedures.

Next, students played with a static worked-example program on the iPad (see Figure 4 for example)**.** This static program presented students with expressions and asked them to simplify the expressions, recording their steps and answers using a Stylus pen (left). Once they entered their solution, the program showed the student's answer next to a correct worked example of the problem (right). This feedback allowed students to compare their answer to a correct solution and was provided regardless of whether they got the problem correct or incorrect. Once students had received the feedback, they moved onto the next problem. The 120 problems in this static lesson were identical to the 120 problems in the dynamic lesson. It is important to note that this static lesson was not designed to be an exact match for the dynamic lesson, but rather to be a high-quality lesson on its own terms that did not use motion and covered the same content.

Measures

Simplifying Expressions Assessments. Each child completed an 18-item assessment involving expression simplification at four different time points: a pretest before instruction, a test after day 1 of instruction, a test after day 2 of instruction, and a retention test 1 month later. These tests assessed two major types of expression-related problem-solving skills: procedural facility with simplification (16-items), and expression transfer (2 items). The problems on the pretest, Day 1 test, Day 2 post-test, and retention tests were identical in form and difficulty but varied with regards to the specific numbers and variables used.

Assessments were coded for accuracy and error analyses on each item were conducted. Each item on the assessment was coded as incorrect, correct, or did not attempt. To determine the source of the error, the following error codes were used: 1) no error, 2) structural error; or 3) addition or negative error (See Figure 5). Since the PS framework is designed to make structure concrete, structural errors are particularly interesting for analysis. Structural errors include combining unlike terms, over-combination (simplifying the expression correctly and then combining un-like terms), and partial structural errors (moving around like terms but not completely simplifying the problem). Addition and negative errors were coded when students used correct structure, but made an arithmetic error when combining terms. When a problem was left blank, it was coded as “did not attempt”. On average, students did not attempt to solve 28% of the pretest problems, 10% of the problems on day 1, 5% of the problems on day 2, and 5% of the problems on the retention test. Further, negative or addition errors were rare, occurring less than 5% of the time.

Mathematics Self-Efficacy and Anxiety Questionnaire. Students were administered a set of 10-items pertaining to their self-efficacy and anxiety in mathematics. To assess students' math self-efficacy beliefs, 5 items were adapted from the Academic Efficacy subscale of the Patterns of Adaptive Learning Scales (Midgley et al., 2000) (e.g., "I know I can learn the skills taught in math this year") (α=.82). To measure students' feelings of math anxiety, 5 items were adapted from the Student Beliefs about Mathematics Survey (Kaya, 2008) (e.g., "I feel nervous when I do math because I think it's too hard") (α=.69). Students were asked to rate how much they agreed with each item on a scale from 0-100 (0= never, 100= all of the time). Scores for each construct were then averaged to create a mean math self-efficacy and mean mathematics anxiety composite (ranging from 0-100).

Student Engagement in Mathematics Questionnaire. Student engagement during mathematics class was measured using a student reported questionnaire on the iPad at 2 time points (day 1 and day 2). 18 items (α=.87) were adapted from the Student Engagement in Mathematics Questionnaire (Kong, Wong, & Lam, 2003): (e.g., "Today I only paid attention in math when it was interesting."). All 18 items were on a scale from 0-100 (0= no, not at all true, 100=yes, very true). Scores from the 18 items were then averaged to create a mean student engagement composite ranging from 0-100 (0=not engaged, 100=always engaged).

Approach to Analysis

First, descriptive statistics and correlations were calculated to determine means and variability for each variable and relations between each construct. Next, t-tests were used to determine whether there were mean differences in achievement between conditions at baseline. Third, we conducted hierarchical regression analyses to examine the relative contribution of variables in predicting structural performance at three different time points. Model 1 predicted performance after day 1 of instruction, after controlling for gender, math self-efficacy, math anxiety, engagement on day 1, and pre-test achievement. Model 2 predicted structural performance after day 2 of instruction. We included the following variables in the analysis: gender, math self-efficacy, math anxiety, engagement on days 1 and 2, pre-test achievement, and achievement after day 1. In the second model, we tested our hypothesis that the concreteness fading condition (dynamic first) predicted improved learning over the concreteness introduction condition (static first). The third model tested whether these effects were retained one month later.

Results

Correlations and descriptive statistics for all variables and outcomes are presented in Table 1. Mean performance by assessment and condition are presented in Figure 6. T-tests indicate that there were no significant differences between conditions at baseline on any predictor variables including gender, math anxiety, math efficacy, or engagement on day 1 and 2 (all p values>0.10) (Table 2). Further, there were no group differences in performance at pretest,

$t(95) = -0.78$, $p=0.44$. At pretest, most students had little to no understanding of like terms and simplifying expressions (on average, only 14% of problems were structurally solved correctly).

Models 1 and 2: Fading vs. Introduction of PS

The regression results predicting performance after Day 1 (Model 1) and day 2 (Model 2) of the intervention, and at retention (Model 5) are presented in Table 3. After the first day of instruction, no significant differences in achievement between those who received the dynamic lesson first or the static lesson first were found, $p=0.61$. Students in both groups made significant gains in their understanding of simplifying expressions; on average, students in both groups correctly simplified 66% of the expressions (Improvement of 52%), suggesting that both lessons were of high quality. Student's prior knowledge significantly contributed to mathematical performance after day 1 ($p<0.01$, $r= 0.41$). Further, math anxiety negatively contributed to student performance after day 1 ($p<0.01$, $r= -0.30$).

Model 2 predicted performance after 2 days of instruction. This outcome is of particular interest because it represents the time point when students had received both the static and the motion based lesson and allows us to compare whether a dynamic first or static first ordering is more effective. While descriptively, students in both conditions improved their understanding of algebraic expressions over the two-day intervention, regression analyses in Model 2 demonstrate a significant main effect for group $F(8, 75) = 7.35$, $p < .01$, $\Delta R^2 = 0.04$, suggesting an order benefit of receiving the dynamic first lesson before the static lesson. Students who received the dynamic lesson first performed approximately 1/4 of a standard deviation higher than the students who received the static lesson first on the Day 2 post-test, after controlling for gender, efficacy, anxiety, pretest scores, engagement, and achievement after Day 1 (effect size=0.22). In the concreteness-fading group, 87.3% of problems attempted were solved correctly without structural errors (*M*=15.71, *SD*= 2.49), while only 74.6% of problems attempted were solved without structural errors (*M*=13.43, *SD*= 4.24) in the concreteness introduction group. Further, performance after day 1 of instruction predicted student performance after day 2; however, pretest scores were no longer significant. More specifically, for every 1-point increase in performance after day 1, performance on day 2 increased by 0.56 points (approximately 3/5 of a standard deviation), after controlling for gender, pretest scores, and engagement on day 1. Math self-efficacy and engagement did not significantly predict performance at day 2.

Model 3: Does this Order Effect Remain at Retention?

Retention assessments 1 month later revealed that although students retained a level of mastery for simplifying expressions, no group differences were observed at retention (p=0.48). However, performance after day 2, as well as math self-efficacy predicted higher retention. A post-hoc analysis suggests that the relatively greater decline in the dynamic-first group seemed to result from a bimodal pattern in forgetting across students (see Figure 7). As can be seen, most students showed a pattern of little to no forgetting across this interval, but three students showed dramatically lower scores on the final test compared to the test after day 2: all three were in the Dynamic First condition. For example, one student in the dynamic first group who got all 18 problems correct on day 2 only attempted to complete the first 7 problems on the retention test (they left the remaining 11 problems blank). One possibility is that while the learning benefits of the dynamic-first order do last over time, the ordering also negatively impacted motivation for later paper-and-pencil post-test in a few students. However, in this study, it is not possible to determine whether the lack of difference at retention is due to true forgetting or some other mechanism (such as lack of motivation).

Discussion

Mathematical formalisms are a productive source for object-based metaphors. In this study, we compared two intervention trajectories within the context of an algebra intervention: fading and introducing the concrete reification of the object-based metaphors present in the manipulation of simple expressions with linear variables. Both dynamic and static lessons were effective in improving student task performance. However, the trajectory that faded dynamic and concrete motion cues led to more robust performance after 2 days than either single lesson and than the static to dynamic ordering (concreteness introduction). We interpret this experiment as a window into studying (and minor support for) our speculation, laid out earlier, that mathematical notation succeeds in part by offloading sentential reasoning into perceptual-motor actions taken over notation.

Prior work examining concreteness in symbols has yielded ambiguous results. Studies that create concreteness through examples that model mathematical principles (Goldstone & Sakamoto, 2003; Goldstone & Son, 2005) have often found advantages for early concreteness. However, our study did not evoke extra-formal domain content in either condition. Other studies which have aimed at rendering concrete intra-formal relations or procedures differentially concrete have either found mixed results, or found an advantage for an abstraction-first ordering

(Kaminski et al., 2008; Kirshner & Awtry, 2004). One key difference between these approaches and ours is that prior experiments have generally involved the introduction of fundamentally new notations, often with distinct properties that facilitated specific strategies.

While prior work has primarily conceptualized concreteness in terms of the utility of bringing in concrete, real-world models (Barab et al., 2007; Goldstone & Son, 2005; Nathan, 2012), many ‘hands on’ technologies are instead about making more concrete the manipulation of symbolic elements themselves (Marshall, 2007). As a hypothetical instance, if a concrete notation represents fractions with pizza slices such that counting the pizza slices is a viable strategy, and students adopt that procedure easily, then they may have trouble transferring to an abstraction that lacks countable objects (Kaminski & Sloutsky, 2012), making concreteness fading unhelpful. However, in our case the distinct affordances of the concrete system—moving and grouping symbols—are the very procedures that are useful in the abstract context. The procedures were made more available in the concrete version, but are fundamentally identical in both. We suggest that in general, concrete notations are likely helpful when they increase the availability of generally useful procedures and understandings that applies across notation systems, and interfere when they suggest or afford procedures unavailable in the dominant abstract notation. We find that, as with more traditional concreteness approaches, initially hands-on tools can facilitate the subsequent learning from static versions--more than the static tools facilitate hands-on experiences. In this case, one possibility is that the fading condition introduced the concepts in a manner that least loaded verbal resources and working memory, and therefore left students better able to learn from future instruction.

Another possibility is that the Pushing Symbols motion condition suggested distinct strategies. Some students who might have relied on memorizing rules may have instead engaged in perceptual-motor visualizations suggested by the notation. Math anxiety has a particular impact on high-working memory strategies such as rule memorization (Ramirez, Gunderson, Levine, & Beilock, 2013). Thus, it may be that part of the advantage of concreteness fading found here results from making available to students on day 1 dynamic (non-working memory based) strategies that can then be deployed in the abstract static intervention.

A critical issue in the design of instructional experiences is that of *cognitive load* (Sweller, 1994), the degree to which (or the ways in which) memory, attention, and reasoning systems are taxed by particular tasks or learning environments. In general, learning suffers when cognitive load experienced during a task surpasses available capacity. The theoretical framework of neural reuse has particular implications for notation design in its relationship to cognitive load. Neural reuse (as discussed earlier), presumes that advanced reasoning occurs to through the cooption of previously domain-specific routines and functions for new purposes. For instance, the system initially developed to detect and recognize distinctive edge patterns in visual input may have been coopted by the development of high-contrast, edge-like writing systems (Changizi, et al., 2006; Dehaene et al., 2015). Similarly, visual notations coopt perceptual-motor routines for processing motion and grouping in real-world objects (Landy, Allen, & Zednik, 2014). This suggests that part of the value of a cleverly constructed notation is that it shifts processing loads away from memory, attention, and reasoning into powerful, high-capacity perceptual and perceptual-motor systems, lowering overall cognitive load. On this account, a virtue of the dynamic lesson is that it provided a context for offloading operations into perceptual-motor instantiations, facilitating additional learning from static notation. On the other hand, the day 1 instructions in the static lesson did not provide a strong context for offloading, so that learning was not as well facilitated on the second day. If this is correct, it suggests that an ideal situation is one that combines an abstract experience with a background history that supports perceptual-motor instantiation of operations—a pattern that has also been used to explain the effectiveness of gesture in mathematics learning (Goldin-Meadow et al., 2001). A secondary but important value is that they demonstrate correct transformations. A student observing their own transformations is likely to see many examples of mistaken rule applications; since the system only allows for correct actions, using the PS system affords the opportunity for passive observation of correct transformations.

This same speculative interpretation has a natural articulation in terms of preparation for future learning (Bransford & Schwartz, 1999). Dynamic symbols, to the degree that they serve as external artifacts that can be internalized as perceptual-motor routines, may provide contexts that facilitate reflection and discovery, both by reducing cognitive load and by providing compelling initial experiences of structure. That is, beyond reducing cognitive load, a perceptual-routine may serve, itself, as the target of reasoning (that is, one can wonder *why* this routine, rather than another, is appropriate). It may be that the strength of the fading condition in part results from the differential value of the dynamic and static conditions for facilitating novel experiences of structure, and reflections on that structure. Relatedly, one possible benefit of drill-and-kill practice is to provide a compelling, fluent, experience of structure that can serve as the foundation for later discovery-based learning (Brunstein, Betts, and Anderson, 2009).

Educational Implications

These results have implications for mathematics teachers and educators and researchers studying mathematical cognition. Most obviously, these results suggest that the dynamic interpretation of algebra can helpfully be presented to students. Both groups benefited from the presentation of dynamical symbols, including the static-first condition that had already received regular symbolic instruction. Simply broadening the number of ways that students are taught to interact with symbols may benefit learning. On the other hand, these results also suggest caution; it seems to matter for learning how and when symbols are treated as static sentences that express meanings, and when and how they are treated as dynamic objects that afford transformation. Surely the full picture of how these approaches trade off with each other remains to be seen, but we can form some tentative hypotheses. First, it seems that many students learned well from each approach: performance after day 1 was already fairly high; but a few students in each group barely improved at all. Second, static approaches to symbols emphasize memorizable rules and can be justified by reasoning about situations; dynamic approaches emphasize learned perceptual patterns, which may be more difficult to explicitly treat as meaningful, or to logically justify by reasoning about situations and models. We speculate that static approaches better situate transformations in meaningful contexts, while dynamic approaches facilitate calculation. Part of the reason may be that treating symbols as objects makes it more difficult to treat them referentially (Kaminski et al, 2006; Uttal, Scudder, & DeLoache, 1997).

Although the actual tradeoffs are likely to be complex, it seems that a good outcome in this case was achieved when dynamical symbols were introduced before meaningful discussions of rules over static symbols. On the surface, this seems to conflict with the speculation above, that static symbols afford justification while dynamic approaches afford rapid computation. It also contrasts with recent findings that Dutch elementary-school students initially construct (presumably rule-based) ways of calculating order of precedence rules that are not sensitive to spatial structure. Sensitivity to spatial properties of symbols (Landy & Goldstone, 2007; 2010) develops over time and expertise (Braithwaite, Goldstone, van der Maas, & Landy, 2016).

This work also adds to a burgeoning literature on the role of technology in the classroom, and more broadly the educational experience of algebra learners. While many mathematics tools emphasize the role of technology in driving motivation and interest (e.g., *Math Blaster*), our work emphasizes a different role for technology: expanding the scope of what can be experienced. Computer technology has played a tremendous role in helping us visualize existing at hard-to-experience scales such as that of an atom or even a cell, or being in impossible places such as circling a black hole; in this case, we are making directly accessible to experience mental operations and actions that had previously to be imagined. Although our language surrounding 'manipulating' equations has involved manual and dynamic spatial elements for centuries, until recently it was difficult or impossible to interact with mathematical formalisms that through their dynamics enforced mathematical laws. Although these 'spaces' are mental, not physical, we find the result to be most comparable to learning tools that simulate these obscure or impossible experiences. Tools that have as a primary goal the facilitation of intra-notational operations are flourishing (popular examples at the moment include Geometer's Sketchpad, Geogebra, and Desmos). These often work by making formal operations physical—but our theoretical understanding of how tools like these might function to transform learning lags somewhat behind. Here, we invoked concepts from distributed and embodied cognition, and neural reuse (Anderson, 2013; 2014) to provide a theoretical account for the value of physicalizing formal operations in dynamic, object-centered interactions.

Consistent with a preparation for future learning perspective (Bransford & Schwartz, 1999), although both groups learned comparably after a single day of instruction—that is, after either type of instruction alone—introducing dynamic symbols prior to static symbols helped students make sense of the static rules in terms of symbolic experiences—literally picking up and moving symbols. This process may have helped them become familiar with the symbolic manipulations, and provided mysteries—experiences that could be made sensible in light of the symbolic rules. We see this kind of sense making as complementary to grounding symbols in related physical experiences (Goldstone, Landy, & Son, 2008). On this perspective, students on their own learn rules first because those are taught first; however in many cases being confronted with symbolic behaviors might help make the rule-construction process more meaningful and better grounded. This perspective is also quite consonant with the idea that procedural, symbolically driven reasoning can be powerful and generative, and is best taught interleaved with more 'conceptual' reasoning processes (Rittle-Johnson, Siegler, & Alibali, 2001).

Limitations and Future Directions

We believe that the primary advantage of perceptual-motor routines comes when they are aligned with transformations. However, we acknowledge that the interface design and gestures used in this study (distribution, for example) may not align perfectly with the intended mathematical transformation. Any interface undergoes iterative development, and is subject to multiple constraints. The version of the technology used in the study reflects a moment in time in the evolution of our gesture-based technology interface, not a firm commitment to a particular mode of interaction.

Although our effects are modest and fade by retention one month later, we believe that these findings are meaningful, especially given that most results showing benefits of concreteness fading have occurred over shorter presentation times and in lab settings. The fact that the effects faded by the retention interval may indicate that the ameliorating effects of object-centered approaches are short-lived, or may be a result of motivation or the mixing between groups over the intervening month. While the retention test provides some indication of overall retention, it may not be a pure measure of group differences. Over the month between the lesson and the retention test, the teacher could have used the instructional strategies from both lessons routinely. Since the students and groups were assigned randomly within-classroom and the teacher was present during the intervention, people with both orderings communicated freely with each other, students received a mixture of different instructional techniques, and there were ample opportunities for teachers to differentially help students who struggled on mathematical concepts related to our lesson, it would be somewhat surprising if the effects did not diminish in strength. However, as such, the findings reported in this study provide only suggestive evidence of concreteness fading, and must be interpreted cautiously rather than conclusively.

We expected the concrete lesson to better engage core cognitive processes. In line with this prediction, students overwhelmingly reported that they enjoyed solving problems and felt like they learned more using PS compared to static and traditional instruction. However, in this study it cannot be definitively suggested that invoking explicit motion increased student engagement. Although student reported engagement was slightly higher in the Pushing Symbols intervention, both on the first day and the second day, these differences were not statistically significant. One potential explanation for this is that both lessons involved solving problems on the iPad, which compared to traditional worksheets is novel and more engaging. Also, the static lesson used iPad pens to record their answers, which were particularly exciting and new for many students. There were also differences between conditions that could have contributed to student engagement. While the PS lesson had somewhat more color, uniquely gave a marker ("coins") of success, and had explicit moment-by-moment feedback, the static lesson gave clearer feedback about procedures through worked examples. It is plausible that enforcing students' use of feedback and error correctly may be more helpful in the beginning of instruction and can be faded out later, but a reverse order may not be as effective. However, while it is conceivable that these differences in feedback could lead to differences in outcomes, it is important to note that the effect of direct feedback was likely low, since both groups learned the same amount from day 1 of instruction. In future studies, there is a need to disentangle the various differences between the conditions. For example, testing static only and concrete only conditions could provide a comparison that could help better understand the benefits or disadvantages to more concrete verses abstract instruction.

Conclusion

Examinations of algebra learning have largely been rooted, necessarily, in counterintuitive notation systems whose mastery involves explicit memorization of rules with minimal perceptual support. These results provide a preliminary demonstration of the possibility of basic algebra lessons that align axiomatic algebraic content, explicit goals, and perceptual and motor activity to yield substantial learning. It calls for careful investigation of possible strategies for learning formal algebras that are intrinsically physical.

More broadly, this research fits into a large collection of recent research emphasizing the importance of the specific use of spatial and perceptual factors in helping children learn reading (Correll et al., 2012; Glenberg, 2011; Kaminski & Sloutsky, 2013). Learners must work with the representations they are shown and create. It is thus intuitive that presentation elements that seem minor to an expert are critical for effective learning, and indeed increasing evidence indicates that very small differences in surface form can have substantive effects on learning outcomes; because these differences can have cascading effects, such formatting issues may have meaningful effects on overall mathematics learning. Symbolic systems are not just tools for expressing meaning—the symbols we use extend the thoughts we can think (Clark, 1998; Iverson, 2007). They do so best when they mesh with pre-existing cognitive and perceptual systems.

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Table 1. Descriptive Statistics and Correlations for All Variables

	1	2	3	4	5	6	7	8	9	10
1. Group (1=Concreteness Fading)	-									
2. Gender (1=male)	0.14	-								
3. Math Anxiety	0.03	-0.11	-							
4. Efficacy	-0.17	-0.03	-0.46**	-						
5. Pretest Achievement	0.08	0.11	0.07	-0.09	-					
6. Day 1 Engagement	0.08	-0.1	-0.39	0.45**	0.12	-				

		4	**							
7. Day 1 Achievement	-0.02	-0.10	-0.20**	0.04	0.40**	0.22**	-			
8. Day 2 Engagement	-0.06	-0.24*	-0.23*	0.24*	0.05	0.62**	0.17	-		
9. Post-test after Day 2	0.29**	0.04	-0.14	-0.04	0.19	0.12	0.54**	-0.02	-	
10. Retention Achievement	0.14	0.01	-0.20*	0.22*	0.19	0.29**	0.50**	0.13	0.77**	-
Mean	0.37	0.44	40.30	75.60	2.27	71.49	11.96	71.90	14.26	14.47
Standard Deviation	0.48	0.5	19.8	17.6	2.50	15.3	4.36	17.	3.85	4.4

		0	0	9		5		00		7
N	98	98	92	92	97	87	93	93	94	94
Note. $^{**}p < 0.01$, $^{*}p < 0.05$										

Table 2 Descriptives and t-test results by Condition

	Concreteness-Fading		Concreteness Introduction					
Measure	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	*Cohen's d*
Math Anxiety	40.95	19.63	39.92	20.06	90	-0.24	0.81	0.05
Efficacy	71.76	19.36	77.85	16.39	90	1.61	0.11	-0.34
Pretest Performance	2.53	3.16	2.11	2.03	95	-0.78	0.44	0.16
Day 1 Engagement	73.14	11.62	70.57	17.10	85	-0.75	0.46	0.18

Day 1 Performance	11.85	4.85	12.02	4.10	91	0.17	0.86	-0.04
Day 2 Engagement	70.59	16.79	72.66	17.23	91	0.57	0.57	-0.12
Post-test Performance after Day 2	15.71	2.49	13.43	4.25	92	-2.85	0.01	0.65
Retention Performance	15.26	4.12	14.02	4.62	92	-1.31	0.20	0.28

Table 3 Summary of Hierarchical Regression for Variables Predicting Algebraic Expression Performance (N=98).

	Model 1- Day 1 Performance			Model 2- Day 2 Performance			Model 3- Retention Performance		
	B	SE	β	B	SE	β	B	SE	β
Intercept	15.32	3.30		11.22	3.07		-2.45	2.93	
Gender (1=male)	-1.75	0.87	-0.21	0.02	0.71	0.00	-0.17	0.62	-0.02
Math Efficacy	-0.03	0.03	-0.10	-0.02	0.02	-0.11	0.04*	0.02	0.20*
Math	-	0.	-	-	0.	-	0.0	0.	0.0

Anxiety	0.07**	03	0.30**	0.02	02	0.12	1	02	3
Pretest Performance	0.70**	0.17	0.41**	-0.03	0.15	0.04	0.05	0.13	0.03
Day 1 Engagement	0.01	0.03	0.02	-0.02	0.03	-0.07	0.01	0.03	0.05
Day 1 Performance				0.56**	0.09	0.63**	0.16	0.10	0.17
Day 2 Engagement				0.00	0.03	-0.01	0.01	0.02	0.05
Day 2 Performance							0.69**	0.11	0.66**

Group (1=Concreteness Fading)	-0.46	0.90	-0.53	1.69**	0.73	0.22**	-0.47	0.66	-0.06
R^2	0.51			0.47			0.65		
F for change in R^2									

Note. **p<0.01, *p<0.05. Model 1 predicts performance receiving one day of instruction. Model 2 predicts performance after receiving both the static and dynamic lessons. Model 3 predicts retention one month later.

Figure 1: An example mathematical derivation. The panel on the left shows the basic transformations as typically represented; that on the right illustrates the dynamic processes involved. Distribution as a perceptual-motor routine involves a ‘splitting’ followed by a ‘translation routine, for example. Red arrows represent the action or movement and purple arrows represent the mathematical transformation.

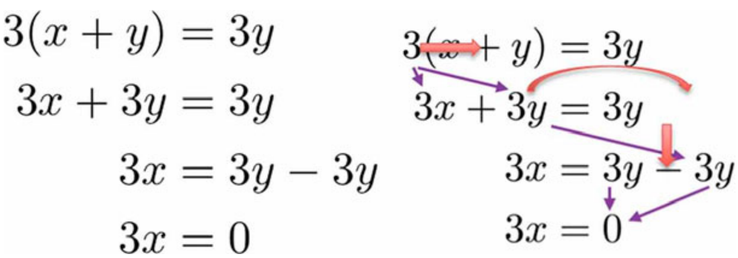

Figure 2: Annotated rendering of the dynamic motion in the technology instantiation of Pushing Symbols. The purple dashed curves indicate user actions; the blue lines indicate responses of the system to the user action. The top row illustrates commutativity; the bottom row indicates factoring. A video of the interaction can be seen at https://www.youtube.com/watch?v=MMVUDTwZmc4.

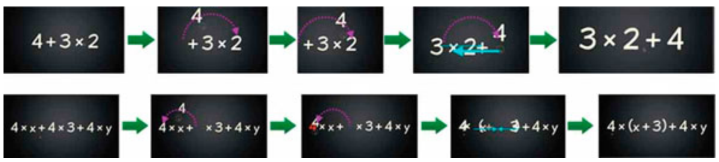

Figure 3. Example of the Pushing Symbols dynamic tile activity.

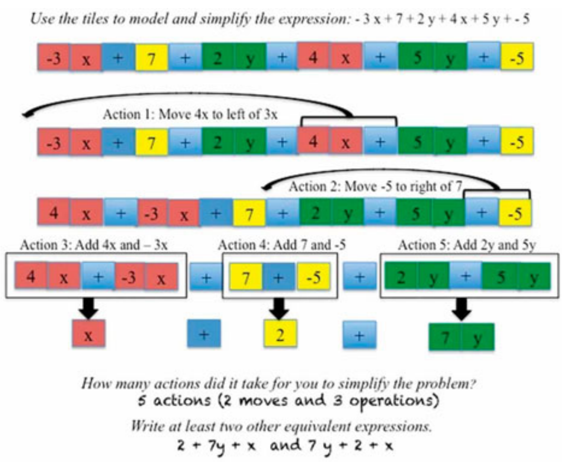

Figure 4. Screenshots of the static iPad worked example program components. Students were initially presented with an expression and asked to simplify. A stylus pen was used to record their work. They entered their final answer on the bottom of the screen. (left) A new screen appeared (right) that compared their answer to a worked example of the problem.

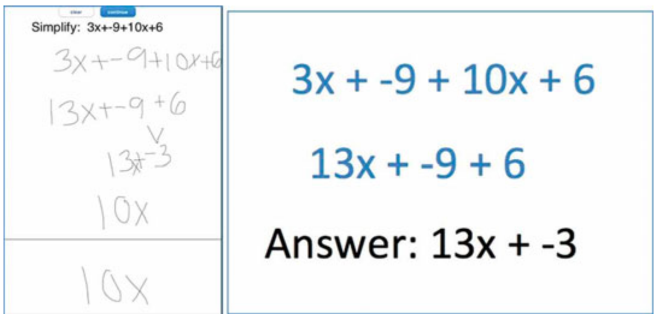

Figure 5. Error types and examples

Error Types		Problem	Example Answer
No Error		4x + 2y + 5 + – 2x + 6	2x + 2y + 11
Structural Error	Full structural error		15x
	Partial structural error		11 + 4x + 2y + - 2x
	Overcombination		15
Arithmetic Error	Addition		2x + 2y + 10
	Negative Errors		-6x + 2y + 11

Figure 6. Mathematics performance on algebraic expressions by condition and time

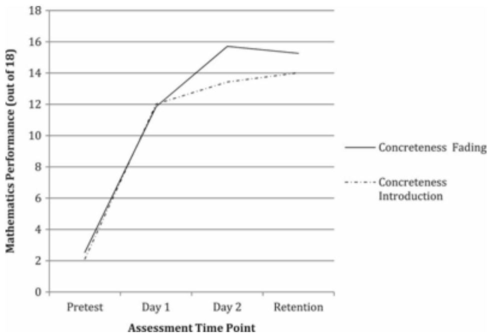

Figure 7. Examination of Achievement After Day 2 and Retention by condition