Toward a physics of equations

David Landy

Department of Psychology, University of Richmond, Richmond, VA 23173
dlandy@richmond.edu

Abstract. Papers on diagrammatic reasoning often begin by dividing marks on paper into two basic classes: diagrams and sentences. While endorsing the perspective that a reasoning episode can be diagrammatic or sentential, I will give an overview of recent evidence suggesting that apparently symbolic expressions in algebra and arithmetic are frequently treated as diagrammatic or even pictorial depictions of objects and events—events that occur not in the content of the expression, but within the notation itself. This evidence suggests that algebra is sometimes less a matter of rules and abstract syntax, and more a matter of constraints on the physical behavior and part-whole structure of notational *things*: an idiosyncratic notational physics, whose laws constrain the structure of proofs. These considerations suggest that whether some marks are a diagram depends on exactly how a user engages them.

Keywords: mathematical cognition, experimental psychology, reasoning

1 Introduction

Our understanding of diagrams often begins from a division of external representation schemes into diagrams and sentences [1,2]. Pictures, blueprints, and maps serve as prototypes of diagrams; the default sentential representation is spoken language. Although modern mathematical expressions superficially resemble quintessential diagrams in that they are typically set off in their own physical space, and use two-dimensional physical space (e.g., in subscripts and superscripts), expressions exhibit many properties typical to sentential schemes. The central contention of this paper is that mathematical forms can be profitably viewed both as sentences and diagrams, depending on how they are used to support reasoning.

It is widely agreed that the content domain of a representation scheme does not determine whether or not it is diagrammatic. As an example, statements of propositional logic may be expressed either with words or with Euler diagrams. A more common perspective is that the relationship between form and content fixes the status of a representation scheme [1,2]. On these accounts, diagrams depict content-level relationships through a homomorphism to physical structures. In sentential systems, physical relationships such as ordering may have a homomorphic relationship to the abstract grammatical structure of a proposition, but not to meaning in the depicted content. Whether a system is sentential or diagrammatic has entirely to do with the system's relationship to its intended content; the user of the system does not contribute to the distinction.

In this paper, I suggest that categorizing representations in this way glosses the actual psychological processes employed by reasoners in solving problems. In particular, in some cases reasoners treat an external representation *as though* it was depicting something that it normatively isn't. This paper provides an overview of recent evidence collected by myself and others, demonstrating that low-level perceptual features of mathematical expressions have a substantial impact on reasoning. Previously, I have argued that reasoning with notations involves the development of specialized perceptual mechanisms [3,4]. Here, I develop an alternative interpretation (see also [5]): Learning mathematical rules involves learning a kind of commonsense physics—the physics of mathematical objects. That is, people often apply to mathematical forms reasoning processes which they typically apply to physical objects undergoing various kinds of change and motion. On this latter interpretation, although there is no homomorphism between the form of a mathematical expression and its normative content, there is an iconic relationship between the surface form of an expression and the representation of symbols as physical objects. The result is a diagrammatic relationship between the physical structure of an expression, and the expression as conceived by the reasoner.

2 Mathematical Expressions as Physical Objects

People know a lot about physical objects. We have a fairly good understanding of how objects move, collide, bend, and break [6]. We also have rich mechanisms for recognizing object boundaries—segmenting visual scenes into objects and their parts.

Infants exhibit knowledge of and interest in the way that objects move, change, appear, and vanish [7-9]. As children explore their environments, they also develop an understanding of which features cue object boundaries [7,10]. By the time they are adults, human reasoners have a rich and developed ontology of different object types, with different kinds of properties. In the same manner that children may initially apply general principles of object segmentation and motion, and over time learn appropriate particular rules for particular kinds of objects [7,8], reasoners learning mathematical systems may adapt general segmentation and dynamic event processes to suit the structure of mathematical expressions. Causally potent experience with objects and affordances shapes children's understanding of specialized situations and objects [8-10]; in a similar manner, causally potent experience with mathematical computations may lead to the incorporation into general physical understandings of constraints particularly suitable to mathematics.

At the very least, people occasionally talk about notations as though they were objects: in Britain, for instance, improper fractions such as $\frac{17}{5}$ are often called "top heavy," suggesting a metaphor to an object standing upright in gravity. Talk of equations as "balanced" suggests similar implicitly gravitational considerations. People often talk of equation solution in terms of motion. Pilot work in my lab found that when asked to describe how to solve generic linear equations, approximately 10-15% of subjects spontaneously described the process of isolating the variable being solved for by using the word “move," and an additional 10% used language suggesting motion. If such descriptions are not purely metaphorical, but indicate processes used in actual online computation, one straightforward hypothesis is that by and large, physical models of mathematics map symbols into material forms by using object segmentation to implement grammatical rules, and understand axioms using representation systems which apply to dynamic events.

2.1 Material approaches to formal grammars

What are the objects that populate the world of algebraic notations? A natural guess is that the structural part-whole segmentation mirrors the formal grammar: objects are expressions, whose parts are connectives and sub-expressions. For example, the expression $9+6\times 7$ is one object, made up of three parts: 9, $+$, and 6×7. The last part is then itself a compound object, made up of 6, $\times$, and 7.

Visual grouping principles require very little training to account for mathematical behaviors. This is because the visual structure of algebraic notations already largely aligns spatial and syntactic proximity[11]. For example, consider the expression

$$3x^2 + \frac{x+\sqrt{4+7x}}{(3+y)x}\ . \qquad \textbf{(1)}$$

The division sign forms a vertical barrier paralleling the syntactic separation into numerator and denominator. Parentheses form a perceptual region, visually grouping the terms within. The overbar in the radical also creates a visually connected region (and is itself the vestige of an obsolete grouping system [12]). Exponents are placed very close to their bases, and omission of the multiplication sign causes products to be spaced more closely than sums. However, some mathematically meaningful segments are not handled appropriately by domain-general grouping principles. In order to correctly group simple arithmetic expressions in uniformly spaced typefaces, such as $3+5\times 4=23$, it is necessary to visually group terms surrounding multiplications preferentially over additions, and those preferentially over equals signs.

An amodal account of expression parsing that relies strictly on rules expressed formally to determine structure provides no particular predictions about the physical requirements of a mathematical system, beyond that the symbols be clearly readable, and close enough that the next symbol can be seen before the previous one is forgotten [13]. However, if the implementation of the rules of interpretation comprises learning idiosyncratic grouping and segmentation principles layered over the usual grouping processes that apply to physical scenes, then a basic biconditional prediction follows: physical features that affect object segmentation should influence the computation of formal syntax, and physical features that influence formal syntax should influence segmentation.

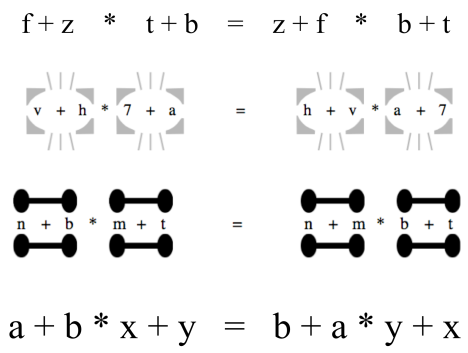

Fig. 1. Sample stimuli from Landy & Goldstone [3] illustrating the effect of (from top to bottom) physical space, common region, connectedness, and alphabetic proximity. In each case, participants were biased to see visually grouped objects as syntactically bound.

Substantial evidence supports the former of these two conclusions. Kirshner [11] demonstrated that students learning a novel arithmetic notation incorporated spatial proximity into syntactic operations. Subjects were better able to respect the order of operations while performing arithmetic expressions, when the novel notation contained spatial proximity relations similar to that of typical algebraic expressions (that is, when higher precedence operations were closer). Landy & Goldstone [3] demonstrated that the effect was not limited either to spacing, or to features present in standard notations. A wide variety of grouping principles affect both perceptual grouping and mathematical competence (see Figure 1).

The implication that mathematical relations should influence spatial perception and grouping of mathematically relevant objects is relatively unexplored (though see [14]). The conceptualization of formal learning presented here makes emphatically the prediction that learning syntax should affect object segmentation.

2.2 Mathematical Rules as Constraints on Physical Change

Once one has found the objects, one must understand how they behave. To be useful, the laws of dynamics must, of course, by and large guarantee mathematically valid results. Many valid manipulations can be accomplished by assuming that expressions are semi-rigid physical forms, with parts that move continuously and that can be created and destroyed in specific kinds of ways and circumstances.

Let's consider two ways to solve linear equations, one using a sentential approach, the other a material. Table 1 presents one derivation of the solution to $y \times 3 + 2 = 8$, using Euclid's axioms. These axioms specify a family of equations—including each of the equations in Table 1—that have the same solution. The key to this method is to apply the rules to find a member of the family whose answer is obvious.

The sentential approach treats the derivation as a sequence of separate statements from the same set (in the case of linear equations, equations with a fixed solution). A short proof or derivation is similar to a paragraph: it is a sequence of separate

Table 1. A sentential approach to equation solving.

Statement	Justification
$y \times 3 + 2 = 8$	Given
$y \times 3 + 2 - 2 = 8 - 2$	Apply Axiom of addition
$y \times 3 = 8 - 2$	Arithmetic Simplification
$\frac{y \times 3}{3} = \frac{8-2}{3}$	Apply axiom of division
$y = \frac{8-2}{3}$	Arithmetic simplification

statements that follow closely upon each other, but which consist themselves of wholly separate words and phrases. Each sentence is a separate thing.

Alternatively, one can see the proof structure as a narrative of transformation, in which one or a few physical objects undergo a succession of alterations. Consider the proof shown in Table 2. It is identical to Table 1, except that two steps have been collapsed. However, the justification is quite different. Here the solver conceptualizes a single equation, undergoing physical transformations. It is unambiguous that there is a single equation, which appears in different forms in the three proof lines as a result of the changes it has undergone (the 2 has "moved rightward" and "changed sign"). Note that this is a kind of motion specific to mathematics; when an object crosses an equation boundary, it must transform (by changing sign).

Consistent with the idea that people sometimes solve problems by treating notations as though they represented motion, Landy & Goldstone [15] found that people solving linear equations were systematically affected by the simultaneous perception of actual motion (see Figure 2). When irrelevant dots in the background behind a problem moved in the same direction that the terms would be moved in the motion-based strategy, error rates were lower than when the dot motion was opposite to that implied by the equation. Furthermore, this effect grew larger with increased mathematical experience, and was strongest on problems that were most familiar (those involving addition and multiplications, rather than subtractions and divisions), suggesting that experience leads to increased use of motion-based conceptions in notations. This is consistent with the hypothesis that situated experience with the physical contingencies of mathematical proofs drives the construction, in reasoners,

Table 2. A material approach to equation solving.

Statement	Justification
$y \times 3 + 2 = 8$	Given
$y \times 3 = 8 - 2$	Move 2 rightward (and change sign)
$y = \frac{8-2}{3}$	Move 3 rightward (and change sign)

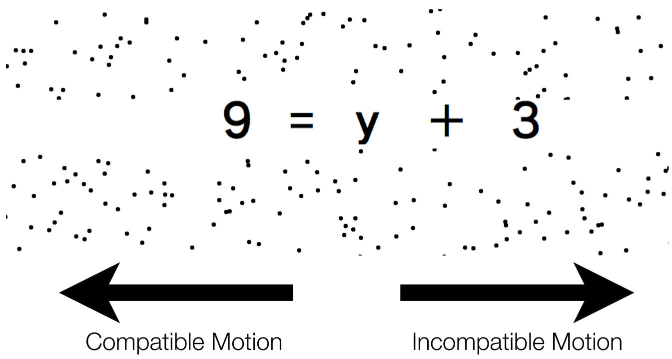

Fig. 2. Sample stimulus from [15]. In the stimulus dots moved quasi-randomly. For this problem, compatible motion is leftward motion; incompatible motion is rightward.

of physical representations of formal systems.

Although we initially described Table 1 using an sentential approach, there is also a readily available material interpretation. Rather than seeing the lines as separate statements, with derivational links justified by the application of rules (the most formally sound perspective), one can see the lines of the proof as dynamic events happening, again, to a single object. In this case, the events would not be motion events, but instead involve creation and destruction of terms. So between lines three and four of Table 1, a "divided by 3" is introduced on both sides of the equation. This language mirrors the usual justification for the axiom itself ("likes done to likes yield likes"); however, in the material approach, the introduced term is not semantically de-referenced. We do not take away two-thirds of each of two piles of stones; we instead insert a pair of symbols "/3." Thus, the justification is intrinsically tied to the notation display, rather than to an underlying situation model. Although I know of no direct evidence that people in fact engage in *this* kind of symbolic-material reification, the material approach predicts that people do. This approach therefore suggests particular phenomena. For instance, semantic and visual facts that prime creation and cancellation should encourage corresponding computational processes and vice versa.

3 Discussion

So what kind of representations are mathematical expressions? When taken to be representations of underlying situations or abstract facts, an abstract syntax mediates physical form and meaning. At least some of the time, however, it appears that the actual mechanisms involved in notation manipulation treat notations *as though* they were literal depictions of physical objects. In these cases, the physical properties of expressions are iconic representations of literal physical objects, and are therefore quintessentially diagrammatic rather than sentential. These considerations suggest a dual treatment of mathematical forms. As depictions of objects, forms can be segmented, manipulated, created, destroyed, and simply observed. As referential symbol tokens, forms can be unpacked into general, meaningful statements. Together, these two systems yield powerful, domain-general syntactic computation.

Not every sentential representation is pictorial in the sense discussed here. The physical interpretation of formal languages makes them different from written natural language. Although proximity may well play a role in sentential understanding, there is no reason to suspect that dynamic transformations of notations play a significant role in sentence understanding. The dual interpretability of mathematical systems may constitute a basic virtue of our modern symbolic systems.

All of this can be summarized as a fairly trivial point about diagrams: diagrams often work by letting one kind of thinking you're good at stand in for a another kind of thinking you're bad at, as when Venn diagrams allow one to come to conclusions about set relations by thinking about spatial relations. Similarly, mathematical expressions written in our modern notation often let you come to conclusions about formal statements and proofs by thinking, instead, about objects in space.

References

1. Chandrasekaran, B.: What makes a bunch of marks a diagrammatic representation, and another bunch a sentential representation? Proceedings of the AAAI Spring Symposium (2005).
2. Stenning, K.: Seeing reason: Image and language in learning to think. Oxford University Press, Oxford (2002).
3. Landy, D. & Goldstone, R. L.: How abstract is symbolic thought? Journals of Experimental Psychology: Learning, Memory, and Cognition 23, 720-733 (2007).
4. R.L. Goldstone, D. Landy, & J.Y. Son, The education of perception Topics in Cognitive Science, (in press).
5. Landy, D. & Goldstone, R. L.: Pushing symbols: Perceived motion in algebraic problem solving, (submitted).
6. Proffitt, D. R.: Naïve Physics. The MIT encyclopedia of the cognitive sciences (1999).
7. Wilcox, T. Object individuation: Infants' use of shape, size, pattern, and color. Cognition 72, 125-166 (1999).
8. Baillargeon, R.: How do infants learn about the physical world? Current Directions in Psychological Science 3(5), 133-140 (1994).
9. Needham, A., Barrett, T., & Peterman, K.: A pick-me-up for infants' exploratory skills: Early simulated experiences reaching for objects using 'sticky mittens' enhances young infants' object exploration skills. Infant Behavior and Development 25, 279-295 (2002).
10. Needham, A.: Improvements in object exploration skills may facilitate the development of object segregation in early infancy. Journal of Cognition and Development 1, 131-156 (2000).
11. Kirshner, D.: The visual syntax of algebra. Journal for Research in Mathematics Education 20, 274-287 (1989).
12. Cajori, F.: A History of Mathematical Notations. Dover Publications: New York (1993).
13. Haugeland, J.: Artificial Intelligence: the Very Idea. MIT Press: Cambridge, Mass. (1989)
14. Jansen, A. R., Marriot, K., & Yelland, G. W.: Comprehension of algebraic expressions used by experienced users of mathematics. Quarterly Journal of Experimental Psychology 56A, 3-30 (2003).
15. Landy, D. & Goldstone, R. L.: Pushing symbols. The 31st Annual Conference of the Cognitive Science Society. Amsterdam (2009).