

Running Head: Space & Syntax

Formal notations are diagrams: evidence from a production task

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Abstract

Although a general sense of the magnitude, quantity, or numerosity of objects is common both in untrained people and in animals, the abilities to deal exactly with large quantities and to reason precisely in complex but well-specified situations—to behave formally, that is—are skills unique to people trained in symbolic notations. These symbolic notations employ typically complex, hierarchically embedded structures, which all extant analyses assume are constructed by concatenative, rule-based processes. The primary goal of this article is to establish, using behavioral measures on naturalistic tasks, that some of the same cognitive resources involved in representing spatial relations and proximities are also involved in representing symbolic notations: in short, formal notations are a kind of diagram. We examine self-generated productions in the domains of handwritten arithmetic expressions and typewritten statements in a formal logic. In both tasks, we find substantial evidence for spatial representational schemes even in these highly symbolic domains.

Introduction

It is clear that mathematical equations written in modern notation are, in general, visual forms, and furthermore that they share some properties with diagrammatic or imagistic displays. Equations and mathematical expressions are often set off from main text, use non-standard characters and shapes, and deviate substantially from linearity of symbol placement. Furthermore, evidence indicates that at least some mathematical processing is sensitive to the particular visual form of its presentation notation (Cambell, 1999, McNeil & Alibali, 2004, 2005). Despite these facts, notational mathematical representation is typically considered ‘sentential,’ and placed in opposition with diagrammatic representations in fields as diverse as education (Zazkis, 1996; Stylianou, 2002), philosophy of science (Galison, 1997; Perini, 2006), computer science (Iverson, 1980), and cognitive modeling and problem-solving (Stenning, 2002; Anderson, 2005).

The standard conception of mathematical notation is best understood via Stephen Palmer’s (1978) classic distinction between intrinsic and extrinsic representational schemes. A representation is intrinsic “whenever a representing relation has the same inherent constraints as its represented relation” (p. 271). Line A being shorter than Line B can be intrinsically represented by the representational element that corresponds to A being shorter, taller, brighter, or larger than the element representing B – any relation that is inherently asymmetric and transitive. Representations are extrinsic when their inherent structure is arbitrary. They model the represented world by explicitly building the necessary structure so as to conform to the world.

Palmer argues that analog representations are intrinsic; correspondences and inferences between represented and representing worlds come for free because of their shared intrinsic structure. Propositional representations, including language, logic, and mathematics, are extrinsic, and hence come to represent objects by explicitly establishing relations with whatever structure is needed. The only intrinsic relation necessary to propositions is the left-right concatenation of basic symbols. Although traditionally understood as extrinsic, it is possible that representations in mathematics and logics nonetheless possess intrinsic and analog properties, and it is this possibility that we empirically pursue here. In a separate paper (Landy, Havas, Glenberg, & Goldstone, in preparation) we consider the case for language.

Several authors have tried to characterize this apparent distinction between diagrams on the one hand, and formal equations and language on the other, while recognizing that both are frequently visual and schematic/abstract representational formats (Stenning, 2002). One prominent proposal is that diagrams represent relational structures *directly*, while notations—formal or otherwise—have structural information mediated via rules over individual elements. That is, assuming that the represented domain consists of relation r , holding over objects in a set $\{a\}$, in a directly represented (diagrammatic) representation schema there will be a metric property of the representation \mathbf{R} (such as spatial proximity), such that \mathbf{R} and r act in direct proportion. In an indirectly represented language, r does not correspond to any metric feature of the representation. Instead, $r(a_1, a_2)$ is expressed via a set of rules acting over concatenative strings—figures in which the only relevant property of a display is the order in which terms appear.

Specifically, we propose that formal notations are diagrammatic as well as sentential, and that the property conventionally described as syntactic structure is cognitively mediated, *in part*, by spatial information. Elements of expressions are “bound” together through perceptual grouping—often induced by simple spatial proximity. Thus, our claim is that mathematical formalizations of syntax are not themselves the direct cognitive mechanisms typically employed in processing that structure. The former really are concatenative, but we propose that people use space and spatial relationships in representational schemas to facilitate the processing of syntax. We are not claiming here that the execution of each individual step in a proof or computation is inherently spatial or processed exclusively using sensori-motor mechanisms. We do suggest that spatial reasoning over the physical layout of notational forms is common in reasoning with formal languages, and that spacing practices play a significant role in human reasoning using notations.

We have argued previously that a broadly similar interference of metric (non-order-related) spatial properties on syntactic judgments provides evidence that spatial processes and representations implement syntax in typical human judgments (Landy & Goldstone, submitted; see also Kirshner, 1989; Kirshner & Awtry, 2004). To study the influence of perceptual grouping on mathematical reasoning, we gave undergraduate participants a task to judge whether an algebraic equality was necessarily true. The equalities were designed to test their ability to apply the order of precedence of operations rules (multiplication precedes addition). Although our participants know these rules, we were interested in whether perceptual and form-based groupings would be able to override their general knowledge of the order of precedence rules. We tested this by having grouping factors either consistent or inconsistent with order of precedence. For example, a participant might be asked whether “ $n * w + y * b$ ” is necessarily equal to “ $y * b +$

$n * w$.” In this example, the physical spacing around the operators is consistent with the order of operators — large spacing around the plus and small spacing around the multiplications. On other trials, the spacings were inconsistent; for instance, participants might be asked to judge whether “ $m+p * e+g$ ” is equal to “ $e+g * m+p$ ” (it isn’t). When physical spacing was inconsistent with order of precedence rules, six times as many errors were made relative to when the spacing was consistent (neutral spacing trials were intermediate in accuracy). Participants continued to show large influences of spatial grouping on equation verification even though they received trial-by-trial feedback. This suggests that sensitivity to grouping is automatic or at least resistant to strategic, feedback-dependent control processes. Even for algebra, one of the clearest cases of widespread symbolic reasoning, spacing exerts a strong and lasting influence.

The fact that people are sensitive to small changes in the physical spacing of formal expressions suggests that symbol systems themselves are grounded, in part, through spatial and visual structures. If so, then people might also respond spatially to the syntax of internally represented expressions, leading symbolic productions to reflect syntactic structure, e.g., the more tightly two mental terms are bound syntactically, the closer they should be written physically. A relationship between spatial and syntactic proximity would be at best unexplained if space is not part of how we represent formal syntax. Models of formal reasoning do not generally relate spatial properties and syntax (indeed, most definitions of formal reasoning overtly exclude visuo-spatial relations), and do not predict any correspondence between physical space and syntax (see in particular Anderson, 2005, Johnson-Laird, 1983). Nevertheless, if, as we suggest, understanding formal symbol structures typically involves spatial resources, then symbolic productions might be expected to reflect syntactic structure: the less tightly two adjacent terms are

bound syntactically, the farther apart they should be placed physically. Study 1 directly examines our proposal by measuring physical inter-operand spacing in handwritten equations constructed by participants from presented word equations.

Experiment 1

In this study, participants were asked to write out simple equations by hand. If, as we propose, formal notations automatically encode spatial relations corresponding to structural relations, then spacing in handwritten equations should reflect the formal structure of the equation. In particular, equality spacing should be very large, since equality signs denote, in all cases, the broadest partition of the sentence. Within the 2-operator side of each equation, spacing should depend on the structure of the expression. In mixed expressions, the middle term is syntactically “bound” to the higher-order multiplication sign, and so the spacing around that sign should be compressed (or that around the lower-order addition sign expanded) relative to its unmixed spacing. Thus, we expect a dependency of operator spacing on the interaction between operator and context.

Although traditional sentential accounts of notational reasoning provide no reason to expect operator spacing to ever be non-uniform, experience with typeset or handwritten equations might drive some kinds of spacing regularities. Typeset equations are generally not fixed-width, and multiplication symbols are generally narrower than addition signs. Experience with typeset equations could lead equation writers to generally space multiplicands more narrowly than addends. However, no prominent equation typesetter adjusts the spacing of terms based on syntax (and if one did, we would regard this as evidence favoring our view!): therefore, while either the spatial encoding or the amodal perspective might predict a main effect of operator, neither the

width of the operators themselves nor experience with typeset equations could lead to the predicted interaction. Thus, the principle theoretical measure is the interaction between operator sign and mixed/unmixed structure.

Method

Participants

24 Indiana University undergraduates participated in the experiment, which fulfilled a partial course requirement. This experiment lasted about 25 minutes.

Procedure

Word equations were presented one at a time on a computer; participants wrote out corresponding equations using standard mathematical symbols. Participants were instructed to use only standard Arabic numerals and formal operator symbols (‘+’, ‘x’, ‘=’), and were explicitly asked not to use any parentheses. Participants were not asked to solve or evaluate the correctness of any equation, nor were they reminded of the correct order of operations.

For each participant, 10 triples of numbers between 2 and 9 were randomly generated (because 1 is much narrower than other numbers, it was excluded from this experiment)

Syntactic binding between terms was systematically manipulated by altering the operator of equations. Each triple appeared in 4 different equations, one with each of the operator structures plus-plus, plus-times, times-plus, and times-times, making 40 translations in all. The other side of the equation contained the same expression, but with the first operation completed. Thus, if the number triple was {2,4,9}, the equations would be $6 + 9 = 2 + 4 + 9$, $2 + 36 = 2 + 4 \times 9$, $8 + 9 = 2$

$x 4 + 9$, and $8 x 9 = 2 x 4 x 9$. The middle two equations are labeled *mixed* operator conditions, while the first and last are termed *unmixed* (since there is no particular hierarchical structure on either side of the equation). In 5 of the 10 equation sets, the triple appeared as the right-hand side of the equation; in the other 5, the triple appeared on the left. This procedure eliminates any interference of particular number choices, since each production is compared to productions that are identical except for operator context.

Each participant received a different, randomly generated stimulus set. Word equations were presented on a computer screen, one at a time, and remained on the screen while participants wrote the corresponding symbolic equation in a printed box (1.1cm high by 10.4cm wide, see Figure 1) on a piece of paper. Word equations employed number words along with the words “times,” “plus,” and “equals”. For instance, if the word equation probe was “six plus five times four equals two plus nine times three,” participants would respond by writing “ $6 + 5 x 4 = 2 + 9 x 3$ ”. Each participant viewed 40 equations in total. Responses that were left blank, contained parentheses or other extraneous marks, or contained crossed out values or other errors were dropped from the analysis. The paper responses were scanned, and the inter-operand spacing between the symbols in the remaining equations were measured using Adobe Photoshop.

We use *operand* throughout this paper to refer to the two numerals on either side on an operator, regardless of operator precedence. Thus, in the expression $1+2x3$, 1 and 2 are for us the operands of the addition (the *addends*); normal mathematical usage would make 1 and $2x3$ the operands. In addition, we will occasionally use the neologism *equalands* to refer to the numerals bounding an equals sign). Although equality is not usually thought of as an operator, it marks the

highest level division of an equation, and so for our purposes serves a syntactic role akin to a very low order operation.

Results

For each participant, spacing was averaged across the stimuli in each condition. The mean values across participants for each context are shown in Table 1. These mean values were analyzed using a 2-way within-participants ANOVA, using distance as a dependent measure, and operator and expression structure as independent categorical variables. As predicted by the typesetting hypothesis, the ANOVA revealed a main effect of operator type: multiplicands were spaced more closely than addends, ($F(2, 46)=7.9$, $MSE=3.35$, $p<0.01$), and equalands were spaced substantially more widely than either ($F(2,46)=105.7$, $MSE=249.2$, $p<0.001$). The interaction between operator type and expression structure was also significant: participants’ compression of multiplicands relative to addends increased in the mixed condition ($F(2,46)=4.726$, $MSE=1.28$, $p<0.05$).

Discussion

The most immediate conclusion that can be drawn from Experiment 1 is that syntax in arithmetic equations is processed automatically. The simple transcription task participants performed in this experiment requires no consideration of syntax at all, but the results show a modulation of productions in virtue of syntactic structure. This is somewhat to the side of our main focus, but is interesting in its own right.

More interesting for our purposes is that writers are not swayed arbitrarily by syntax—they construct spatial properties which match their own perceptual biases. Terms were spaced more narrowly when they were grouped more closely. Historical interactions with typeset equations do not predict these effects, nor do traditional symbolic accounts of mathematical competence. So of the hypotheses considered earlier, this result is compatible only with the suggestion that participants systematically vary spacing according to the particular syntactic structure of the current equation. Because this behavior presumably generalizes to the population at large, including the teachers and parents of our particular participants, historical interactions with other hand-written equations could account for the results—participants in our task might reasonably be sensitive to syntax because their teachers and parents were, and therefore sensitivity of spacing to syntax forms part of the participants' training. That is, our participants may have received more training with mathematical expressions in which the spacing is consistent with the syntactic structure. However, this explanation does not provide any additional insight into why this spacing convention has been adopted in the first place. For that, the most parsimonious account for the environmental regularity is, once again, that spatial processes are involved in the representation of mathematical syntax in the normal course of algebraic reasoning.

There is a plausible alternative to the spatial information hypothesis: it might be that syntax processing mediates (somehow) access to the lexicographic forms for numbers and symbols. If syntactically bound items are chunked in memory, for instance, then access to terms within a chunk may be quicker than access to terms across chunks (Cheng & Rojas-Anaya, 2006). If so, and if horizontal pen movement between characters correlates with access time (if for instance the pen is moved at some more-or-less fixed velocity while the lexical form is being accessed), then a

memory delay could produce increased spacing. Experiment 2 addresses this possibility by exploring spacing behavior on a typed input task.

Experiment 2

A limitation with Experiment 1 is that the formal system used was a small (though important) one: pre-algebra using equality, addition and multiplication. While this system is convenient in that it is widely known and studied, nevertheless such a small system with several idiosyncratic features makes it difficult to generalize the results. Experiment 2 broadens the scope of our examinations by exploring a very different notational system: formal propositional (quantified and unquantified propositional) logic. In addition, Experiment 2 uses a very different experimental format, and a very different response generation system on the part of the participants; instead of asking participants to write unused and useless pseudo-equations in a laboratory setting, Experiment 2 involves a corpus analysis of self-generated expressions by participants interacting with a Web-based teaching tool designed and maintained by Colin Allen and Chris Menzel, and based on an accompanying textbook (Allen & Hand, 2001).

Method

Materials

The current analysis is based on the "Logic Daemon and Quizmaster" (Allen & Menzel, 2006, <http://logic.tamu.edu/>) which is an interactive Website designed for use with the Logic Primer textbook by Allen & Hand (2001). Students can use this Website to tackle exercises found in the textbook as well as additional problems of the same type. For our analysis, we chose to focus on the 'translation' exercises which require students to render sentences of English into the

formal system described in chapters one (propositional) and three (predicate logic) of Logic Primer. In these exercises, students are presented with up to five English sentences and under each sentence is a standard Web form single-line text input field. Students freely type a response for one or more of the sentences and click a submit button. For instance, problem 9 from problem section 1.3 (in chapter 1) states that “If Mary dances although John is not happy, Bill will dance”. Instructions state that “Q”, “S”, and “R” are to be used to denote the atomic sentences “Mary dances”, “John is happy”, and “Bill dances”, respectively. Participants enter a formal sentence corresponding to this sentiment (one correct response would be “ $((Q \ \& \ \sim S) \rightarrow R)$ ”).

Each string of characters submitted in this way is checked first to see whether it represents a well-formed formula (wff) according to the specifications of the formal system. Although the textbook uses non-ASCII characters, these are mapped to ASCII strings for keyboard input; specifically, the single arrow is represented as '>' (dash-greater-than, double arrow with '<>' (less-than-dash-greater-than), the upside-down 'A' of universal quantification with '@', and the backwards 'E' of existential quantification with 'S'. The wff formation rules specify the use of parentheses around binary sentential connectives: '&' (and), 'v' (or), '>' (if...then), and '<>' (if and only if). Some of these parentheses may also be omitted following a formal convention that is defined in chapter one of the text. The parenthesis-dropping conventions follow the specified order of operations: & and v precede >, and > precedes <>. Any string that passes the wff test is next checked for correctness with respect to the particular translation problem attempted (logical equivalency to a stored answer. Both the wff check and the correctness check is indifferent to any white space introduced by the student, and when problems are returned to students with feedback, any spaces are removed.

It is worth noting that although interface, formal system, physical situation, and participant pool and motivation are different from Experiment 1, the task is quite similar: in both cases, participants are asked to take a natural language statement and translate it into a formal system.

Analysis

129,526 submissions to the translation verification interface (exercises 1.3 and 3.2) submitted between May 5, 2005 and April 4, 2006 were collected and analyzed. These raw submissions were reduced in several ways. First, the initial interface prompts a user to submit several translations at once; as a result, many users submit a single typewritten response many times, while revising other sentences on the same page. In order to eliminate repeated entries, repetitions of literally identical responses submitted from a single IP address on a particular day were eliminated. Also, many responses violated rules of the formal system that were neither syntactic nor semantic, e.g., using incorrect symbols (e.g., “=>” instead of “>”, “and” instead of “&”, or “V” instead of “v”), failing to translate by simply retyping the English text prompt, etc. Our overriding interest is in the relationship between syntax and space, so submissions with invalid operator symbols were removed. Finally, many submissions were simple junk, e.g. “asdf”, “dfdfdfdf”. These were also removed. After all of these reductions, 48,131 statements from 595 unique IP address/time stamp combinations remained.

The translation verification interface automatically determines the well-formedness of an expression following the formal syntax. If the expression is well-formed, then the interface determines whether it forms a correct answer to that problem. Both of these pieces of information are provided to the submitter, who then has the option to revise the submission. The same

verification system used to provide submission feedback was used to categorize submissions for our analysis.

We distinguish three physical spacing conditions: spacing *consistent* with the operator structure, spacing *inconsistent* with operator structure, and no spacing at all (*unspaced*). An expression is considered consistent when the space around every operator in the expression is appropriate: spaces around conjunction, disjunction, conditional, and bi-conditional signs should be even, spaces should only be inserted to the left of negation signs and quantifiers. If any spacing violated these constraints, then the expression was flagged as inconsistent.

Our predictions are as follows: first, we predict that because representations of space play a role in the way reasoners process syntax, participants using the site will at least occasionally insert spaces. Although random insertion of spaces would be far more likely to produce inconsistent than consistent spacings, we predict that .spacing will be primarily consistent, and that only consistent spacing will improve performance.

Experiment 1 indicated that spacing is modulated in the presence of hierarchical syntax: on this basis we predict that participants will be more likely to produce spaces in responses containing more than one operator. Because more skilled reasoners are less likely to be dependent on perceptual support (Chi, Feltovich & Glaser, 1981), we also predict that more advanced participants will be less likely to space expressions at all. The problems studied came from two sections of the book; one on propositional logic, and one on predicate logic. Our prediction then is that spacing will be more common in the first, more elementary section. Finally, because the spatial representation theory implies that consistent spacing is indicative of deep syntactic processing, we predict that accuracy will be higher on consistently spaced

statements than on other statements, whenever structure matters (i.e., whenever there are two or more operators).

In analyzing these data, we do not attempt to evaluate the statistical significance of our results for two reasons: first, the breakdown of submissions by unique IP address/date does not adequately divide submission into independent samples. Since we have no way to determine unique individuals, and furthermore no way to determine the relationship between individuals, statistical tests based on the assumption of independent samples are inappropriate. Second, the large size of the sample guarantees that standard statistical measures will indicate significance (all of the contrasts considered here are highly significant by standard measures), regardless of the underlying mechanisms. For these reasons, we report frequencies without invalid statistical measures.

Results

Table 2 presents the frequencies of submission broken down by spacing, number of operators, and logic type. As expected, participants frequently spaced expressions. 10.8% of all expressions submitted contained some spacing. When expressions were spaced, moreover, they were predominately consistently spaced: 82.6% of all spaced equations were consistent with operator syntax. Consistently spaced submissions were also more likely than either inconsistent or unspaced expressions to be correct; 53% of consistent equations were correct, compared with only 37% of inconsistently spaced and 50% of unspaced equations.

In order to test the structure sensitivity of consistent spacing, we divided the dataset according to whether a problem required syntax resolution (that is, whether it had two or more connectives, see Table 3). Participants did indeed space more frequently on multi-operator

problems (9.2% of multi-operator problems were consistently spaced against 6.8% of few-operator problems). Furthermore, accuracy was highest (53%) when expressions were consistently spaced, and lowest on those inconsistently spaced (37%; 50.1% of all unspaced expressions were correct), principally on multi-operator expressions. On few-operator problems, 52% of consistent, 42% of inconsistent, and 52% of unspaced submissions were correct. However, the number of problems is very small (there were only 26 inconsistently spaced few-operator submissions in all).

We also tested the theory that more training would reduce the need for formally extraneous spacing. Translation problems appear in two sections of the textbook: Chapter 1 (propositional logic) and Chapter 3 (predicate logic). We divided the full dataset into these two categories, and measured spacing frequency across these two categories (see Table 4); both consistent and inconsistent spacings were more frequent on propositional problems (13.1% and 2.4%, respectively) than on predicate logic problems (6.7% and 1.6%).

Discussion

Despite being formally unnecessary and informally discouraged, spaces were frequently inserted into typed sentences of formal logic. These spacings were nearly always consistent with the operations they abutted; submissions with consistent spacing were also slightly more likely to be correct than unspaced submissions. Together with Experiment 1, Experiment 2 establishes that people working in two very different domains systematically space formal systems that formally do not require differential spacing.

Because the participants in this experiment were typing on a keyboard, a chunking account that predicts differential spacing on handwritten equations as a result of differential chunking in memory cannot account for spacing here. The most plausible theory that predicts systematic spacing in both of these domains is that syntax processing is bound up with spatial representation.

Other accounts of the results of Experiment 2 are possible, however. For instance, other formal systems, such as programming languages, are often taught with explicit instructions to space logical terms; participants might be transferring this practice from programming experience. This possibility cannot be definitively eliminated (though why spacing is common in programming is *still* mysterious), the fact that more experienced reasoners space less, not more, seems incompatible with the idea that appropriately spacing is an acquired skill.

General Discussion

In both typed logic and handwritten arithmetic translation tasks, participants created formally irrelevant spatial relationships in stimuli. In both cases, these relationships aligned with the syntactic structure of the formal statement being expressed. The kinds of regularities produced in Experiment 1 have been shown to benefit correct syntactic interpretation (Landy & Goldstone, submitted), and those of Experiment 2 seem to be at least correlated with higher accuracy submissions. People seem to spontaneously create alignments of space and syntax that, when perceived, help them reason formally.

While behavioral evidence cannot conclusively establish that syntactic structure in mathematics is generally *represented* using spatial representational systems, it does establish a processing connection between space and syntax that is difficult to otherwise explain. Regardless of the internal structure of syntactic representations, the evident relationship between metric

spatial properties and syntactic behavior indicates that the latter is, in Stenning's sense, a directly interpreted property of mathematical equations. Mathematical & logical forms often literally function as diagrams, serving to ground the abstract relationships they express in more immediately available sensorimotor relationships. In the normal course of events, spatial grounding of syntax guarantees spatial alignment with the judgments that purely concatenative rules demand.

Our claim has been that external representations of syntactic systems carry formal information in their spatial properties. This is the converse of other arguments that mathematical reasoning involves visuo-spatial processing rather than formal operations. For example, Hadamard (1949) has complained that the true heart of a mathematical proof -- the intuitive, often visual, conceptualization -- is ignored in the formal description of the proof steps themselves. The mismatch between how mathematics is formally presented and how it is concretely conceived by practitioners has been often noted (Lakoff & Nuñez, 2000). Our claims complement these analyses because both this and our approach seek to integrate perceptual processing and mathematical reasoning. However, our claim is that even when we consider symbolic transformations, visuo-spatial processes are still pertinent and strongly influential. Thus, unlike some proponents of embodied cognition (Wilensky & Reisman, 2006), we do not advocate drawing a sharp division between perceptual and formal routes to mathematical understanding. We believe that formal routes to mathematical understanding are perceptual!

Space has been proposed as a mediating metaphor for metric properties such as time (Boroditsky, 2000, Casasanto & Boroditsky, 2003), numerosity (Lakoff & Nunez, 2000; Dehaene, S., Bossini, S., & Giraux, P., 1993; Fischer, 2001, 2003), pitch (Casasanto, Philips, & Boroditsky,

2003) and many others (Casasanto & Lozano, 2006). Behavioral evidence typically taken as evidence for the spatial processing of temporal reasoning has included priming effects of spatial experience on temporal interpretations of spatial/temporal language (Boroditsky, 2000). Another traditional source of evidence for the role of spatial representations is the spontaneous production of systematic spatial behaviors that are not task-appropriate. For instance, Casasanto & Lozano (2006) observed participants as they told pre-constructed stories about events which contained non-spatial concepts for which spatial metaphors were common, e.g. that a particular stock increased in value ("went up"). Even when the language speakers used did not contain a spatial metaphor, participants often used spatial gestures which were compatible with the typical metaphor, indicating that the spatial metaphor was mediating their explanations. One possible explanation of our results is that representations of syntax are typically derived metaphorically from representations of space, just as those of time or numerosity seem to be. We are not certain that the data support such a strong conclusion. Our primary claim is that the rule-systems governing interpretation of formal systems carry functional spatial information—that they are diagrammatic. Metaphorical spatial derivations are, however, one plausible potential account of spatial representations of syntax.

That spacing facilitates syntax is important for our understanding of mathematics and mathematical learning, but it is also important for education and cognitive psychology more generally. For education, our results suggest increased sensitivity to the physical features of how mathematics is presented to students and how they present mathematics to their teachers. Physical properties such as spacing may be used to give students a perceptual scaffold for the rules underlying algebra. Further research is necessary to know whether these scaffolds, when

removed, help students to continue to obey the appropriate mathematical rules or if they act as crutches that thwart rule development. Reciprocally, by examining students' spacings of their own productions, we may be able to diagnose their misunderstandings. In the same way that manual gestures are sensitive indicators of inchoate explicit mathematical understandings (Alibali & Goldin-Meadow, 1993; Goldin-Meadow, Wein & Chang, 1992), production spacing may indicate the beginnings, or lack thereof, of knowledge for order of precedence.

One possible source of the influence of syntax on space is that mental representations of formal syntax are internally spatial. Another possibility is that internal representations really are amodal, and that the differential spatial proximities between math or logic elements only arise during production. By this account, people have an amodal, extrinsic represented form that resembles standard symbolic conceptions. This representation incorporates syntax explicitly by formal structure rather than spacing. However, when participants produce physical forms in the production tasks, they are influenced by the syntax in their own spacings. This alternative remains a possible account of our results, and further explorations will be required to fix the source of the spatial-syntactic interaction, but there are several considerations that make the production-based account unattractive. First, our earlier results show that perceptual tasks, not just production tasks, are influenced by spacing (Landy & Goldstone, under review). It is possible to explain both sets of results by peripheral changes to perceptual input and motor output processes, but it is more parsimonious to have a unified account of both perception and action in which representations are intrinsically spatial. Second, a related consideration of parsimony leads us to prefer a single representation that includes syntactic and spacing information, rather than positing two representations where these sources of information are separated. We now know

that spacing influences, and is influenced by, syntactic form, and so the burden of proof may be said to be on those who argue that there is a deep syntactic representation that is completely isolatable from visuo-spatial information. Third, it leaves unanswered the question of *why* syntax influences spatial productions, whereas in our account space is part of the represented formalism because it promotes correct syntactical parsing.

Following this third point, one likely reason why our participants' show sensitivity to syntactic form in their physical spacing of equations is that this spacing information promotes their own proper syntactic interpretation of their equations. We asked participants to produce equations without performing any additional operations on them. However, this is atypical. Participants usually write equations out to then solve them. Even when our participants do not have to solve equations, they are apparently using the same production habits that they typically use when reasoning mathematically. Over the course of history, mathematical formalisms have evolved so that they are cognitively helpful devices, and this has entailed making apparently superficial, but practically crucial, form changes. For example, the convention introduced by Descartes (Cajori, 1927) of using letters near the beginning of the alphabet to denote constants those near the end to denote variables frees us from the burden of remembering which are which, and allows us to use our memory for other aspects of a mathematical situation. Similarly, using variable subscripts rather than distinct letters has no mathematical function at all; the only role of subscripts is cognitive: they help a reader understand and remember which variables are related.

Our claim is that this is a case of cognitive ontogeny recapitulating phylogeny. The physical properties of mathematical formalisms are adapted not only across generations, but also within a single person. People presumably learn to incorporate syntactically-consistent spacing into their

physical equations because it helps them obey the formally sanctioned rules of mathematics. Even when these physical adaptations are discouraged (Experiment 2), they persist because people either learn by trial-and-error or intuit by a meta-cognitive understanding of their own math skill that formalisms that incorporate spacing end up serving their needs better.

Most fundamentally, our results challenge conceptions of symbols as amodal and divorced from analog, spatial information. In this respect, we offer a reinterpretation of Newell and Simon's (1963, 1976) influential "Physical Symbol System Hypothesis." Their hypothesis was that physical symbol systems had the necessary and sufficient means for producing intelligent action. A symbol system includes both physical symbols such as marks on paper or punches on a computer tape, and the explicit rules for manipulating these tokens. In action, all of their physical symbols were distantly related to their worldly referents, and were digital and discrete entities such as the strings "P \supset Q" and "GOAL 7 TRANSFORM L3 INTO LO." The arbitrary nature of these entities was by design because they wanted symbols to be able to designate any expression whatsoever without any *a priori* prescriptions or limitations. We concur with Newell and Simon's emphasis on *physical* symbols, and believe in paying even more attention to symbols' physical attributes involving space, shape, and perceptual grouping. Accordingly, our revised physical symbol systems hypothesis is that symbols are not arbitrary, unconstrained tokens, but rather are represented and processed using space and perceptually organized groups. This conception of physical symbols makes them far more constrained than those underlying Newell and Simon's General Problem Solver, but these constraints are not only limiters, but permitters as well. For Specific Problem Solvers that are humans, it is good policy to design symbols that can be processed efficiently given what we know about perceptual and cognitive mechanisms. From

this perspective, it is hardly surprising if the symbols we write look a lot like those that we are good at reading, and if the symbols we think with are a lot like those we are good at thinking.

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Table 1: Mean spacing (with standard error) by measurement condition in Experiment 1 (mm)

Operator	Unmixed	Mixed	Overall
Addition	9.52±.48	9.77±.48	9.65±.48
Multiplication	9.38±.50	9.16±.53	9.27±.51
Equality	12.38±.59	12.13±.57	12.25±.58

Table 2: The use of spacing in typed formal translations.

Type	Logic Type					
	<i>Propositional</i>			<i>First Order</i>		
	U	C	I	U	C	I
<i>Single Op</i>						
Correct	1,344	160	11	1,187	12	0
Incorrect	1,592	145	15	735	9	0
<i>Multiple Ops</i>						
Correct	6,175	1,058	159	12,899	1,044	165
Incorrect	5,182	844	213	13,817	1,011	339

Results from Study 2. Entries specify number of submissions of each category. U denotes unspaced submissions, C those in which spacing and operations were consistent, and I submissions with at least one inconsistently spaced operation.

Figure 1: Sample stimuli from Experiment 1. The first and second stimuli are mixed examples, the third and fourth are unmixed. Each set of operands appeared in all four operator contexts. For this experiment, the distance between operands on the single-operator side were ignored.

11:

$$12 + 2 = 2 \times 6 + 2$$

16:

$$2 + 12 = 2 + 6 \times 2$$

28:

$$12 \times 2 = 2 \times 6 \times 2$$

40:

$$8 + 2 = 2 + 6 + 2$$